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Approximation of the Higher Meson States Mass Spectra X(3872), Y(4140), Z(4430), X(5568)

Scientific research paper

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ABSTRACT

Theoretical research is conducted to determine the mass spectra and wave functions of the higher di-molecular type of meson states $X(3872), Y(4140), Z(4430), X(5568)$ in the Hellmann potential with relativistic effects and corrections. Femtoscopic J/ψ bound states have essential concepts in high-energy physics, especially in creating two-point correlation functions for the hadronic molecule channels that build the T_{cc} state. We analytically calculated the radial part of the Schrödinger equation using the transformation to relativistic space using the normal ordering operators. The excited states of eigenenergy are obtained. The constituent mass of the di-molecular type of mesons was determined, which appears due to relativistic effects. It is informative to describe strong interactions and bound state mechanisms using quantum field theory ideas and quantum mechanics. In conclusion, we provide advancements in the higher meson states mass spectra in the relativistic limit. The results provide satisfaction in comparison with experimental data and the work of other researchers.

1 Introduction

In recent years, higher meson states have been discovered in the Belle experiment in Japan and the BESIII experiment in China at particle accelerators and then quickly confirmed by the Collider Detector at Fermilab experimental collaboration using proton-proton collisions at a center-of-mass energy of $7\,TeV$ [1], BaBar Collaboration Statement in decays of B hadrons at the Tevatron and LHC, and the DΘ short-distance phenomena detector. All the higher meson states are typically named by their mass, such as $X(3872)$. The first new higher state with a mass of $3871.69 MeV$, the quantum number 1^{++} was discovered in 2003. It has a mass of $3872 MeV$ and has been observed in several experiments and can be created directly through the high energy strong interactions of the colliding quarks and gluons. However, after its discovery, the properties of the $X(3872)$ exotic meson are still under deep research. It decays into an upsilon meson and two charged pions incomes via an intermediate K^0 mesons; hence it is a new, excited upsilon meson state. It could be a hadronic molecule, which is a weakly-bound charm-meson molecule (loosely bound of two mesons as a di-molecule), or a di-quark and a di-antiquark (tetra-quark $ccuu$). Only 26% of the production rate is observed from decays of B hadrons (Fig. 1). Whether these molecular suppositions of $X(3872)$ structure, the Collider Detector at Fermilab experimental collaboration measured several properties of $X(3872)$ with higher precision than ever before in 2011 and

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continues to make valuable significant contributions to clarifying the nature of the new exotic charm states.

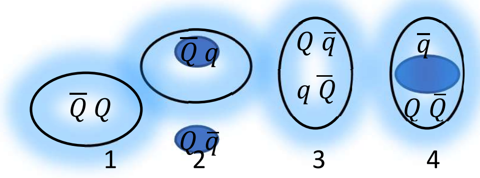

Figure 1. X(3872) in different multi quark models 1-meson, 2-hadronic molecule, 3-tetraquark, 4 hadrocharmonim

Other examples of new higher hadrocharmonim meson states include the $Y(4140)$, the $Z(4430)$, and the $X(5568)$[2-4]. The discovery of these new higher meson states has led to new insights into the nature of strong force and the behavior of quarks inside multi-quark hadronic systems. In the previous research, their exact characteristic and properties are studied. This is while there is too much research in progress to understand better higher meson states and their role in high-energy interactions.

In 2008 the second higher meson state $Y(4140)$, with the mass $4140MeV$, was detected at the KEKB by the Belle collaboration accelerator in Japan and in 2009 at the Fermi National Accelerator Laboratory. It has a large decay width, which suggests that it is not an arbitrary and formal state. Hence, it presents as a hadronic bound state (tetraquark) which is composed of $cc\bar{c}\bar{c}$. Physicists called $Y(4140)$ a charmonium-like state [5]. The higher meson state with the mass $4430MeV$ and the narrow decay width was first observed in 2009 by the collaboration at the Fermilab Tevatron particle accelerator and confirmed in 2014 by the collaboration at LHCb CERN. It contains quarks and gluon which brings us closer to hybrid behaviors. Based on its characteristics, the $Z(4430)$ can present as a hybrid heavy meson or $c\bar{c}d\bar{u}$ with the quantum numbers quantum numbers $J^P = 1^+$making it a tetraquark candidate. Currently, the heaviest meson is $X(5568)$. It was found in 2016 at the Fermilab Tevatron with a large decay width and mass of $5568\,MeV$ and can be understood as a $su\bar{b}\bar{d}$ state. The X(5568) has quantum number $J^P = 0^-$ in the scalar state, $J^P = 1^-$ in the vector state, or $J^P = 2^+$ in the tensor state.

Theoretical and high energy physicists called this higher meson state an exotic hadronic molecule. The hadronic molecule is a multiplex particle of more hadronic bound states by strong interactions. Hence, detecting these higher meson states presents a new insight into the properties of quarks in high-energy physics. Therefore, the higher meson state is the main issue in theoretical and experimental particle physics. From 2015 to 2020, researchers at the Large Hadron Collider at CERN in Switzerland studied the properties of a hadronic molecule, which consists of two mesons. The study on the meson characteristics of bound states at high energy was based. It decayed into J/ψ bound state. Hence, the mass spectrum and eigenenergy of these meson states are very significant in describing the behavior of hadronic states. The mass spectra of the higher meson states can be defined within the foundation of the Schrödinger equation in nonrelativistic quantum mechanics. In the strong interactions, we cannot neglect relativistic effects.

Therefore, we can transform the kinetic energy part in the Schrödinger equation to the relativistic form that we call the semi-relativistic Schrödinger equation. This equation gives us a good mathematically relativistic correction presentation and description of the higher meson states within the Hellman potential $U(r) = -\frac{A_1}{r} + A_2\frac{e^{-\alpha r}}{r}$ [6]. We try to define the actual relativistic corrections to the mass spectra using the framework of quantum field theory because of the small value of the relativistic effect, i.e., we determine and calculate the relativistic corrections to the non-relativistic interaction potential. The selected potential is a type of potential energy used in non-relativistic quantum mechanics to describe the behavior of two interacting particles with the electric charges (The Breit potential type was developed in the 1920s by Gregory Breit). This type of potential describes the relativistic and electrodynamic behaviors of interacting systems, i.e. it determines the spin-spin interactions for the effects of corrections in the relativistic limit and the electrostatics interactions of particles in bound states. The Breit potential is an essential tool for studying the behavior of quarks bound states inside the hadrons. Therefore, the chosen type of potential and Caswell-Lepage idea on the nonrelativistic quantum field theory help us to describe these bound states with the relativistic corrections on mass spectra. Another idea for calculating the mass spectra of hadrons is the Caswell-Lepage method. Caswell and Lepage presented an idea in the quantum field theory at a nonrelativistic limit. In this approach, the Lagrangian operator is described by expansion in powers of the velocity of the constituent particles rather than the coupling constant. The Caswell-Lepage idea focused on the system's behavior in the nonrelativistic limit. Constituent particles are moving much slower than the speed of light, which means that relativistic effects can be systematically neglected. By expanding the Lagrangian in powers of the particle velocity based on the Caswell-Lepage formalism, we can derive a set of effective field theories that describe the behavior of systems at the nonrelativistic limit. Their approach has been instrumental in studying the behavior of heavy multi-quarks states, such as the higher meson states, which are fundamental and principal in the study of hadron physics and also very practical for predicting the wide range of experimental data. Hence, these approaches explain the S-matrix and the behavior of the interactions. In this research, we study the higher meson bound states with rational Feynman path integral in quantum field theory. We determine and calculate mass spectra and energy eigenvalue using the two intertwined spaces within the normal ordering method in quantum mechanics. Hence, observations of spin-exotic meson behavior and properties of systems like tetraquarks and molecular hadronic states can improve precision on molecular meson mass, widths, and decay modes, evidence for hybrid mesons with excited gluonic degrees of freedom. As we know, experimental candidates for exotic bound states of two quarks are very important. For example, a heavy upsilon particle could be a $B^* \bar{B}^*$ system, or X(3872) in different multi quark models can be as a 1-meson, 2-hadronic molecule, 3-tetraquark, 4hadrocharmonim. In experiments, we defined more candidates for exotic hadronic states. Because of the static approximation for some of the quarks' type, their spin and isospin decouple make the pseudoscalar mesons and the vector mesons degenerate, while physically they have a very small value of separation. We measure wavefunctions symmetric under the interchange of the mesons with the quark spin and isospin being singlet or triplet; these then couple to different combinations. So, in summary, while spin and isospin classifications remain foundational, ongoing studies continue advancing our understanding of meson spectra and interactions within and beyond the standard model.

The remainder of this research is laid out in the following manner: Section 2 introduces the bound state formalism in the relativistic limit and quantum field theory using the Feynman path integral. In Section 3 the normal ordering method to calculate the Schrödinger equation and define the mass spectra of the higher meson states

$$X(3872), Y(4140), Z(4430), X(5568)$$

with relativistic corrections. Finally, Section 4 includes concluding remarks.

2 Relativistic formalism to bound states

The mass spectra of the higher meson-bound states from the formal transformed ($E \rightarrow E = \sqrt{p^2 + m^2}$) in the Schrödinger equation with the mathematical calculations is practically impossible. Therefore, the most essential issue in theoretical particle research is to explain the Einsteinian adjustment of higher meson-bound states, in order to determine the characteristics of relativistic effects within the potential interaction and kinetic energy. We present the method based on quantum field theory and Feynman path integral to calculate the mass spectra of hadrons. As we know, the long-range behavior of the propagator function of the related currents with the specific quantum numbers can determine the mass spectra of hadronic bound states. The presentation of the propagator in quantum field theory as a functional integral allows us to average over the external field. This approach is very close to the Feynman functional path integral in the Schrödinger picture in quantum physics, where relativistic effects are not considered. By the side of the path integral, the Feynman diagram determines the interaction potential within the exchange of the mass and the field. The mass exchange of component particles describes the constituent mass value. i.e., the kinetic term of the total Hamiltonian expressed in terms of the constituent mass of the component particles in the hadronic systems. The component particle mass (the rest mass) differs from the constituent mass. The constituent mass presents the relativistic effect of interactions. We show that the constituent mass is important for heavy multi-quark states, such as higher molecular meson-bound states, and is noticeable when compared to the rest mass of the constituent meson. We explain the related current of charged quarks in the multi-quark hadronic state and represent the propagator in the form of the corresponding current by averaging over the field **A** for two bounded mesons [7-10]. This defines the kernel function of two charged mesons with the same or different rest masses. Then we can determine the two-point function by averaging over the field $\Pi(r-r') = \langle G_{m_c} G'_{m_{\bar{c}}} \rangle_{\boldsymbol{A}}$. By the variational method, the two-point function presents in the form of path integral, which is like Feynman's functional in non-relativistic quantum physics. The two-point function and the propagator at the limited distant $x \to 0$ present the Feynman path integral for the motion of particles with masses μ_1, μ_2 in the non-relativistic quantum theory with the $W_{i,j}$ potential interactions. These interactions contribute to the rescaling quark mass. We have an exchange between mesons and field, and the other one with each other. The total potential interaction within the relativistic corrections reads [7]

$$W_{i,j} = \frac{g^2 i^{i+j}}{2} \iint d\tau_1 d\tau_2 Z^i(\tau_1) G\left(Z^i(\tau_1) - Z^j(\tau_2)\right) Z^j(\tau_2) \quad , \tag{1}$$

where the functional integral is over the 4-dimensional spacetime, $G\left(Z^i - Z^j\right)$ is the propagator of the field **A**, $W_{1,1}, W_{2,2}$ is the self-energy of meson interactions, and $W_{1,2}$ is the mesons' interaction with the field **A**. If the molecular meson mass $m_1 = m_2 = m_c$, then the mass spectra of the higher meson bound states are determined as follows [7]

$$M = \min_{\mu} \left(\frac{m_c^2}{\mu_c} + \mu_c + E_\ell(\mu) \right), \tag{2}$$

where $\mu = \frac{1}{2\mu_c}$, $E_\ell(\mu)$ is the eigenenergy of the radial Schrödinger equation $HR(r) = E_\ell(\mu) R(r)$, and μ_c is the constituent mass of mesons, which is a relativistic correction to the meson rest mass and is determined by

$$\mu_c = \left(m_c^2 - 2\mu^2 \frac{\partial}{\partial \mu_c} E_\ell(\mu) \right)^{1/2}, \tag{3}$$

and otherwise $m_1 \neq m_2$ the higher meson-bound states are determined as [7]

$$M = \min_{\mu_1, \mu_2} \left(\frac{m_1^2 \mu_2 + m_2^2 \mu_1}{2\mu_1 \mu_2} + \frac{\mu_1 + \mu_2}{2} + E_\ell(\mu) \right), \tag{4}$$

and

$$\mu_1 = \left(m_1^2 - 2\mu^2 \frac{\partial}{\partial \mu_1} E_\ell(\mu) \right)^{1/2}, \tag{5}$$

$$\mu_2 = \left(m_2^2 - 2\mu^2 \frac{\partial}{\partial \mu_1} E_\ell(\mu) \right)^{1/2}. \tag{6}$$

As a result, we define the mass spectra of the higher molecular meson bound states formalism by considering relativistic correction and relativistic correction to the interaction term included in the two-point function $\Pi(r-r')$.

3 The Schrödinger equation in the normal ordering form

Our start point is creation of the higher $Y(4140)$ meson bound state based on the radial Schrödinger equation ($\hbar = c = 1$) in 3-dimensional space, which describes the interaction of two mesons with the masses $m_1 = m_2$ in the Hellmann potential $U(r) = -(A_1 - A_2) r^{-1} - A_2 \alpha$ which reads

$$\left(-\frac{1}{2\mu} \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right) + \frac{\ell(\ell+1)}{2\mu r^2} - (A_1 - A_2) r^{-1} - A_2 \alpha - E_\ell(\mu) \right) R(r) = 0, \tag{7}$$

where $m_1 = m_2 = m_c$, $\mu_1 = \mu_2 = \mu_c$

$$\frac{1}{\mu}=\frac{1}{\mu_1}+\frac{1}{\mu_2}=\frac{2}{\mu_1}\to\frac{1}{\mu}=\frac{2}{\mu_c}.$$

The Hellmann potential occurs in nature, whereas the harmonic oscillator is an approximate model, which works for small oscillations, but is inappropriate to use to describe anharmonic systems. The Hellmann potential is used to study the diatomic molecules and might be considered as a potential with behavior between exactly solvable harmonic oscillator and nonlinear anharmonic models. Based on the oscillator representation method [7], we describe this interaction and use the harmonic potential as base of a quantum system before describing the other potential as interaction Hamiltonian. Reference [7] completely describes the main idea of the oscillator representation method.

By presenting equation (7) with the new variable

$$r=q^{2\rho},\ \rho\succ 0,\ R(r)\to\psi(q^{2\rho})\ ,$$

the Laplacian gives the new form as

$$\Delta_q=\frac{d^2}{dq^2}+\frac{\mathcal{D}-1}{q}\frac{d}{dq}. \tag{8}$$

By considering

$$\hat{q}=\left[\frac{2m\omega}{\hbar}\right]^{1/2}(\hat{a}^{+}+\hat{a}\,),$$

$$\hat{p}_q=i\left[\frac{m\omega}{2\hbar}\right]^{1/2}(\hat{a}^{+}-\hat{a}\,), \tag{9}$$

where $\hat{a}^{+}$ is the raising operator while $\hat{a}^{-}$ is the lowering operator, in a new axillary space $\mathcal{D}$ as a form of the normal ordering method we then explain the Schrödinger equation (**Error! Reference source not found.**) in the form of canonical variables with the oscillator frequency ω of the bound state due to the higher $Y(4140)$, the meson bound state is a quantum oscillating system [7,11]. In this axillary space, we require that the Hamiltonian interaction does not contain the quadratic form of $\hat{q}^2$, this is a main condition for the normal ordering method in the new axillary simplistic $\mathcal{D}$ space. Using this condition, we obtain the oscillator frequency ω. The interaction term of Hamiltonian in the new form of the canonical operators within the Hellmann potential is obtained as

$$H_0\psi=\frac{\hat{p}_q^2}{2}+4\mu\rho^2q^{4\rho-2}\big(-(A_1-A_2)q^{-2\rho}-A_2\alpha q^{2\rho}-E_\ell(\mu\,)\big)\psi=0. \tag{10}$$

Equation (10) is the interaction Hamiltonian without relativistic interactions of spin-spin, spin-orbit interactions. We can include the interaction between the meson's spins in the $Y(4140)$ meson bound state. Therefore, in the modified Schrödinger equation (Eq. (**Error! Reference source not found.**)), we substitute

$\hat{p}^2=\sqrt{\hat{p}^2+m_c^2}$, and based on Eq. (2)

one can define

$$\sqrt{\hat{p}^2+m^2}\approx\min_{\mu}\tfrac{1}{2}(\mu+\tfrac{\hat{p}^2+m^2}{\mu})$$

and then for the $Y(4140)$, reads

$$\sqrt{\hat{p}^2+m_c^2}\approx\min_{\mu_c}\frac{1}{2}\left(\mu_c+\frac{\hat{p}^2+m_c^2}{\mu_c}\right). \tag{11}$$

We represent the modified Schrödinger equation in the form of the normal ordering described in Eq. (**Error! Reference source not found.**) and then define the total modified Schrödinger equation
with relativistic spin-spin interactions [12]. The total Hamiltonian is

$$H=H_0+H_{SS}+H_{LS}\ ,$$

$$\varepsilon\psi=(H-E_\ell)\psi=0\ ,$$

$$\varepsilon=\varepsilon_0+\varepsilon_{SS}+\varepsilon_{LS}, \tag{12}$$

Where H_0 is the pure Hamiltonian (without spin interactions), H_{SS} is spin-spin interactions part of the Hamiltonian, H_{LS} is spin-orbit interactions term of the Hamiltonian, and reads

$$H_0=\frac{\hat{p}_q^2}{2}+4\mu\rho^2q^{4\rho-2}(A_1-A_2)r^{-1}-A_2\alpha)-4\mu\rho^2q^{4\rho-2}E_\ell(\mu\,),$$

$$H_{SS}=\frac{1}{12\mu_c^2}(S_1\cdot S_2)\Delta_q((A_1-A_2)q^{-2\rho})=\frac{1}{6\mu_c^2}(S_1\cdot S_2)(A_1-A_2)q^{-6\rho},$$

$$H_{LS} = \frac{1}{4q^{2\rho}}\left[\boldsymbol{A}(LS_+) + \boldsymbol{B}(LS_-)\right]\nabla_{q^{2\rho}}((A_1 - A_2)q^{-2\rho}) = \frac{3}{2\mu_c^2}(LS_+)(A_1 - A_2)q^{-6\rho},$$

where

$$\boldsymbol{A} = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} + \frac{4}{\mu_1\mu_2},$$

$$\boldsymbol{B} = \frac{1}{\mu_1^2} - \frac{1}{\mu_2^2}.$$

For the $Y(4140)$ molecular meson-bound state with the constituent meson mass $\mu_1 = \mu_2 = \mu_c$ and spins S_1, S_2, we have $j = \ell + S, S_+ = S_1 + S_2,\ S_- = S_1 - S_2$. S is the eigenvalue of the total spin momentum: $S = \pm 2$ for the parallel direction of spins and $S = 0$ for the antiparallel direction of spins, S_1, S_2 are the eigenvalue of the spin moment of each quark, ℓ is the eigenvalue of the orbit momentum, and j is the eigenvalue of the total angular momentum. To take into account the contribution of the spin interaction, we need to define the spin-spin and spin-orbit scalar products:

$$(S_1 \cdot S_2) = \frac{1}{2}(S(S+1) - S_1(S_1+1) - S_2(S_2+1)),$$

$$(LS) = \frac{1}{2}(j(j+1) - S(S+1) - \ell(\ell+1)).$$

Now we determine $E_\ell(\mu)$ and mass spectra of the $Y(4140)$ meson bound state using Eqs. (**Error! Reference source not found.**) and (**Error! Reference source not found.**). After some mathematical changes, we define

$$\varepsilon\psi = \frac{D\omega}{4} - 4\mu\rho^2 q^{2\rho-2}(A_1 - A_2) - 4\mu\rho^2 q^{4\rho-2}A_2\alpha - 4\mu\rho^2 q^{4\rho-2}E_\ell(\mu) - \frac{2}{3\mu}\rho^2 q^{-2\rho-2}(S_1 \cdot S_2)(A_1 - A_2) + \frac{3}{2\mu}\rho^2 q^{-2\rho-2}(L \cdot S_+)(A_1 - A_2), \tag{13}$$

and after a little simplification of relations, we define

$$\varepsilon_0 = \frac{D\omega}{4} - 4\mu\rho^2 q^{2(\rho-1)}(A_1 - A_2) - 4\mu\rho^2 q^{4\rho-2}A_2\alpha - 4\mu\rho^2 q^{4\rho-2}E_\ell(\mu),$$

$$\varepsilon_{SS} = -\frac{2}{3\mu}\rho^2 q^{-2(\rho+1)}(S_1 \cdot S_2)(A_1 - A_2),$$

$$\varepsilon_{LS} = \frac{3}{2\mu}\rho^2 q^{-2(\rho+1)}(LS_+)(A_1 - A_2). \tag{14}$$

Now we can determine ρ. The parameter ρ used by the variational method is found. This method is a technique used for approximating the lowest energy eigenvalue. By choosing and finding the values of this parameter for which the expectation value of the energy is the lowest possible. The minimum eigen energies value of the molecular meson-bound state $Y(4140)$ is defined from $\frac{d\varepsilon_0(E_\ell)}{d\rho} = 0$, and one can approximate the parameter ρ for the first relation in equation (**Error! Reference source not found.**). The parameter ρ , for quantum harmonic systems, is $0 \prec \rho \preccurlyeq 1$, and for anharmonic quantum systems $2 \preccurlyeq \rho \preccurlyeq 3$. Then by $\frac{d\varepsilon_0(E_{n\ell})}{d\omega} = 0$, we determine ω. Hence, Eq. (**Error! Reference source not found.**) using

$(\rho = 1)\ q^2 = \frac{D}{2\omega}, q^4 = \frac{D(D+2)}{4\omega^2}, q^4 = \frac{D(D+2)(D+4)}{8\omega^3}$ reads

$$\varepsilon\psi = \frac{D\omega}{4} - 4\mu(A_1 - A_2) - 4\mu q^2 A_2\alpha - 4\mu q^2 E_\ell(\mu) + \frac{\Sigma}{\mu}q^{-4}, \tag{15}$$

where

$$\Sigma = \left[-\frac{2}{3}(S_1 \cdot S_2) + \frac{3}{2}(LS_+)\right](A_1 - A_2).$$

Equations $\varepsilon\psi = 0$ and $\frac{d\varepsilon\psi}{d\omega} = 0$, give us the energy eigenvalue of as the $Y(4140)$, the meson-bound state within the modified Hellman potential at the relativistic limit is

$$E_\ell(\mu) = -\frac{\Gamma(2\ell)(A_1 - A_2)\Sigma}{\mu^2\Gamma(2\ell+3)}\omega^3 + \frac{(4\ell+4)\,\Gamma(2\ell+2)}{16\mu\Gamma(2\ell+3)}\omega^2 - \frac{\Gamma(2\ell+2)(A_1-A_2)}{\Gamma(2\ell+3)}\,\omega - A_2\alpha, \quad (16)$$

and the oscillator frequency of the $Y(4140)$ is defined as

$$\frac{3\Gamma(2\ell)(A_1-A_2)\Sigma}{\Gamma(2\ell+3)}\omega^2 - \frac{(4\ell+4)\,\Gamma(2\ell+2)}{8\mu\Gamma(2\ell+3)}\omega - \frac{\Gamma(2\ell+2)(A_1-A_2)}{\Gamma(2\ell+3)} = 0, \quad (17)$$

And hence, if we want to use the approximate form of the wave function or mass spectrum, we can use the values of ρ instead of $\frac{d\varepsilon_0(E_\ell)}{d\rho} = 0$, and define the higher meson bound state for different parameters ρ as follows

$$\omega = \left(\frac{(3+2\ell)\Gamma(\frac{3}{2}+\ell)}{12(A_1-A_2)\Sigma\Gamma(\ell)}\mu\right)^2, \; if \;\; \rho = 1/2$$

$$\omega = \left(\frac{(7+6\;\;)\Gamma(\frac{7}{4}+\frac{3\ell}{2})}{54(A_1-A_2)\Sigma\Gamma(\frac{3\ell}{2})}\mu\right)^{4/3}, \; if \;\; \rho = 3/4$$

Hence, for the minimum energy eigenvalue, the parameter ρ was approximated $\rho = 1$, and the oscillatory frequency of higher meson-bound states reads

$$\omega = \frac{(1+\ell)\Gamma(2\ell+2)}{12(A_1-A_2)\Sigma\Gamma(2\ell)}\mu. \quad (18)$$

Then, spectra for the radially excited states are defined by substituting Eqs. (**Error! Reference source not found.**) and (**Error! Reference source not found.**) into Eq. **Error! Reference source not found.**) which reads

$$M = \min_{\mu}\left(\frac{m_c^2}{2\mu} + 2\mu - \frac{(A_1-A_2)\Sigma}{2}\omega^3 + \frac{1}{8\mu}\omega^2 + \frac{(A_1-A_2)}{2}\,\omega - A_2\alpha\right). \quad (19)$$

Then the constituent mass $\mu_c$of mesons in the higher $Y(4140)$ bound states and the reduced mass μ can be determined. According to Eq. (**Error! Reference source not found.**) we have

$$\frac{dE_\ell(\mu)}{d\mu} = \left[\left(\frac{(1+\ell)\Gamma(2\ell+2)}{24}\right)^2 \frac{1}{2(A_1-A_2)\Sigma\Gamma(2l)} - A_2\alpha\Gamma(2\ell+2)\right]\left[\frac{(1+\ell)\Gamma(2\ell+2)}{12(A_1-A_2)\Sigma\Gamma(2\ell)\Gamma(3+2\ell)}\right], \quad (20)$$

and then we define the reduced mass $\mu = m_c(4 + 2\frac{dE_\ell(\mu)}{d\mu})^{-0.5}$ and the constituent mass of mesons $\mu_c = m_c(1 + \frac{dE_\ell(\mu)}{2d\mu})^{-0.5}$ with the rest meson mass m_c in the higher molecular meson-bound state. Based on the diquark–antidiquark picture, we can calculate the mass spectra of X(3872), Y(4140), Z(4430), and X(5568) with the relativistic correction. We present the mass spectra results for $Y(4140)$, in Table 1 for the higher molecular meson-bound states with quark masses [13]

$m_s = 93.4\ MeV, \qquad m_c = 1.27\ GeV,$

meson's mass [13]

$D^*_s = 2.112 GeV$

and parameters [14]

$A_1 = 1.58\ GeV,\ A_2 = 0.243\ GeV,\ \alpha = 0.325\ GeV.$

and with the quantum number in the triplet $S = 1$ states:

ℓ	$S_1.S_2$	(LS)	Σ
1	1/4	1	0.5
2	1/4	2	1.33

Table 1. Y(4140) mass spectra, the constituent mass of meson, and the oscillator frequency in (GeV).

	ℓ=1		
$\boldsymbol{\rho}$	*0.5*	*0.75*	*1*
$\boldsymbol{M}$	*4.178*	*4.151*	*4.147*
$\boldsymbol{\mu_c}$	*2.080*	*2.108*	*2.110*

$\boldsymbol{\omega}$	0.862	0.645	0.789
	$\boldsymbol{\ell=2}$		
$\boldsymbol{M}$	4.299	4.272	4.520
$\boldsymbol{\mu_c}$	2.100	2.105	2.097
$\boldsymbol{\omega}$	0.57	1.151	4.423

4 Conclusions

The higher mesons X(3872), Y(4140), Z(4430), X(5568) bound states structural properties are described. The Schrödinger equation is investigated in normal ordering form and transformed into a new auxiliary space, which helps us to approximate the mass spectra of bound states within the Hellmann potential, with relativistic corrections. The framework of quantum mechanics, quantum field theory, and quantum chromodynamics provided valuable information on determining relativistic correction to the mass spectra of the higher mesons bound states. The results can open new windows for further theoretical and experimental investigation in the field of particle physics because of the relativistic corrections that we include in calculations. In this study, we used mesonic interaction to describe and approximate the mass spectra of higher molecular mesons bound states in the excited ℓ=1,2 states within a simplistic D -dimension axillary space. We defined that the constituent meson masses are not free parameters; they depend on the rest mass and reduced mass for each constituent particle. We select different values for the parameter $0 \prec \rho \preccurlyeq 1$, and based on this parameter, we can approximate the wave functions of the higher molecular mesons bound states with the oscillator frequency $\omega^{\rho} = \frac{D\Gamma(0.5D+\rho-1)}{48\rho^2(A_1-A_2)\Sigma\Gamma(0.5D-\rho-1)}\mu$ of higher meson-bound states. The oscillator frequency calculated for $l = 1$, at $\rho = 0.5$ is 0.862 GeV, at $\rho = 1$ is 0.789GeV, and for $l = 2$ at $\rho = 0.5$ is 0.57GeV. It presents that the best approximation of minimum mass spectra for the first and second excited states is at $\rho = 1$. As we presented in Table 1 the constituent mass of $Y(4140)$ increases with increasing the excited states. It describes that the relativistic correction directly includes the mass spectra and mass of the particle in the bound states and the mass spectrum of the $Y(4140)$ increase with excited states level. The result is in good agreement with the value 4146.5±4.5+4.6−2.8 MeV [15], the LHCb collaboration experimental data, and 4143.0 ± 2.9 ± 1.2 MeV [15] the Collider Detector at Fermilab experimental collaboration results.

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